Orientations of prolate ellipsoids in general shear flow

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Goals

Final Goal

- Better understand the collision probability of small non-spherical particles in turbulent flows.
- Better understand the contribution of collisions to the growth of ice particles in clouds.

Present Goals

- Study the orientation dynamics of a single particle in general shear flows.
- Study the dynamics of ensemble of orientations - p.d.f in general shear flows.

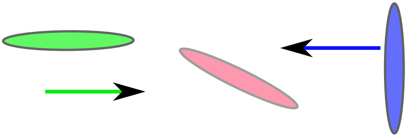
Collision Efficiency

Collision efficiency is determined by both:

- relative velocity
- orientations probability distribution

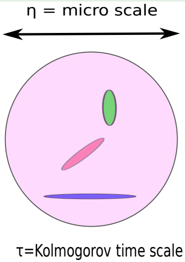
Collision Mechanism

- Particle relative translational motions are induced by eddie's acceleration and particles' mass difference.
- Particle orientation depends on the shear.



Assumptions

- **Particles are smaller than Kolmogorov microscale.**
- **The flow near a particle and its near neighbors may be approximated as a linear shear flow.**
- **This linear flow persists for a time comparable to Kolmogorov time scale.**
- **Mean collision efficiency may be obtained from averaging of many realizations.**



Outline

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- Study the evolution of the orientations of a single particle in a linear shear flow and its relation to the particle shape (aspect ratio).
- Find attractors: fixed points and limit cycles.
- Find characteristic time scale to reach the attractors and compare them with *Kolmogorov time scale.*
- Study the dynamics of the orientation p.d.f of ensemble of initial orientations.

Setup

orientation:

Polar Representation:

$\theta =$ polar angle ;

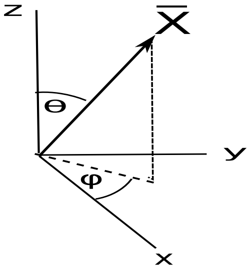
$\phi =$ azimutal angle.

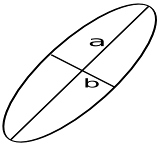
Cartezian Representation:

$X = (x, y, z)$ a point

on the unit sphere.

$x^2 + y^2 + z^2 = 1$





Aspect ratio:

$\beta = \mathbf{b}/\mathbf{a}$

Rotation Equation of Motion

Equation of Motion: Low Reynolds Number

$$\frac{d}{dt}(\mathbf{I} \cdot \omega) = -\mu \mathbf{\Omega} \cdot \omega + \mu \mathbf{N} \tag{1}$$

Small inertia: **Quasi Steady** equation of motion:

$$\omega = \mathbf{\Omega}^{-1} \mathbf{N} \tag{2}$$

The torque N acting on a particle, embedded in a flow field U is

$$N = \mathbf{\Omega} \cdot \tilde{\mathbf{T}} \cdot (\tilde{\Diamond} \times \mathbf{U}) \tag{3}$$

$$\Longrightarrow \omega = \tilde{\mathbf{T}} \cdot (\tilde{\Diamond} \times \mathbf{U}) \tag{4}$$

Linear Flow

Linear Shear Flow

$$U_i \equiv dX_i/dt = S_{ij}X_j; \quad S_{ii} = 0 \tag{5}$$

Decomposing the shear tensor

$$\begin{aligned} S_{ij} &= E_{ij} + R_{ij} \\ E_{ij} &= \tfrac{1}{2}\left[S_{ij} + S_{ji}\right] \\ R_{ik} &= \tfrac{1}{2}\left[S_{ik} - S_{ki}\right] = \epsilon_{ijk}\xi_j \end{aligned} \tag{6}$$

E – Strain Tensor – Deformation

R – Rotation Tensor – $R_{ij}X_j = \overline{\omega} \times \overline{X}$

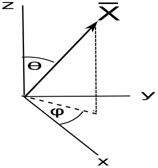
Jeffery Equation: From eqs. (4) & (5) [small inertia]

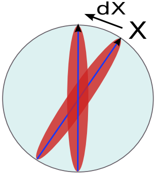
$$\gamma \frac{dx_i}{dt} = J_{ij} x_j - x_i(x, Ex) \equiv \gamma u_i(x)$$

$$J_{ij} = E_{ij} + \gamma R_{ij}$$

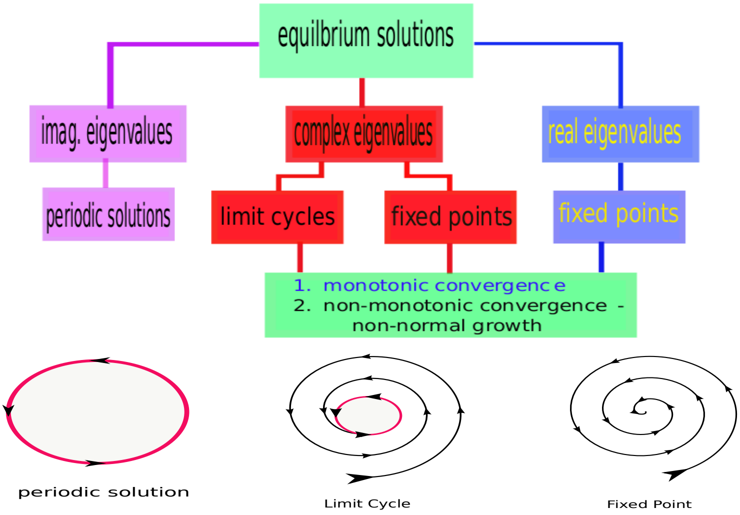
$$\gamma = \frac{1+\beta^2}{1-\beta^2}; \quad 1 < \gamma < \infty$$

Jeffery Equation preserves length: $||x|| = 1$ $\frac{d}{dt}||x|| = 0$





Stationary solutions



Real eigenvalue - fix point

Let X^1 be a normalized eigenvector of J corresponding to a real eigenvalue λ^1 then X^1 is a fixed point.

Jeffery Equation

$$\gamma \frac{dx}{dt} = J \cdot x - x(x, Ex)$$

$$J \cdot x^1 = \lambda^1 x^1$$

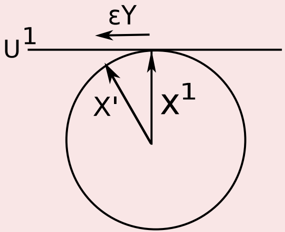
$$x^1(x^1, Ex^1) = x^1(x^1, Jx^1) = \lambda^1 x^1$$

r.h.s. vanishes $\Longrightarrow$ X^1 is a fixed point.

Stability of fixed points - real eigenvalues

return

Perturbation of Jeffery Equation



X^1 - a fixed point; U^1 subspace; $U^1 \perp X^1$

X' - a small deviation from X^1:

$$x' = x^1 + \epsilon y + \dots$$

$$y \in U^1$$

Perturbed Jeffery Equation

$$\gamma \frac{dy}{dt} = Ly - \lambda_1^J y$$

$$Ly = (P_{U^1} \cdot J)y = J \cdot y - x^1(x^1, J \cdot y)$$

- $L : U^1 \rightarrow U^1$ **is a 2-D operator**
- **The two eigenvalues of** L **are the remaining two eigenvalues of J:** $\lambda_{1,2}^L = \lambda_{2,3}^J$
  X^1 **stable** $\Longleftrightarrow \lambda_1^J > \lambda_2^J, \lambda_3^J$
- L **is non-normal** $LL^t \neq L^t L$
  Eigenvectors are not orthogonal.
  Non-monotonic convergence is possible.

return

Non-normal growth

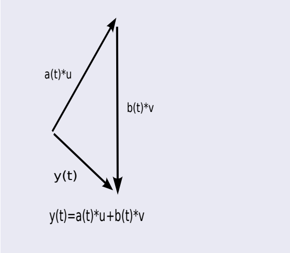
$$\overline{y}(t) = a(t) \cdot \overline{u} + b(t) \cdot \overline{v}$$

$$a(t) = a(0) \exp\left(\frac{\lambda_2^J - \lambda_1^J}{\gamma} \cdot t\right); \quad b(t) = b(0) \exp\left(\frac{\lambda_3^J - \lambda_1^J}{\gamma} \cdot t\right)$$

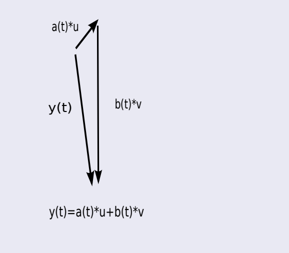
Suppose $\lambda_2^J << \lambda_3^J$

Suppose $(u, v) \neq 0$

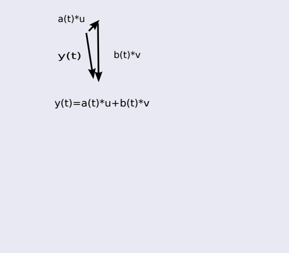
$t = 0$



$t = t_2 > 0$



$t = t_3 > t_2$



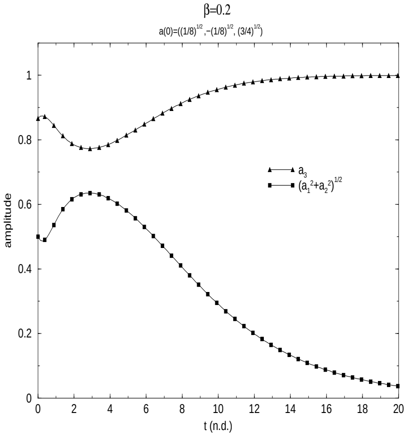
Example I: non-normal growth Contd.

components - parallel and perpendicular to the stable fixed point x^3
a_1, a_2 - perpendicular
a_3 - parallel

non-normal growth

$$S = \begin{pmatrix} -1 & 3. & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}$$

$$\frac{\tau_c}{\tau} = \frac{\gamma}{\tau(\lambda^J_{max} - \lambda^J_{int})} \approx 4.5$$
$$\tau = (E_{ij}E_{ij})^{-1/2} = 0.41$$



return fp-I

Example II: monotonic convergence Contd.

components - parallel and perpendicular to the stable fixed point x^3
$a_1,\ a_2$ - perpendicular
a_3 - parallel

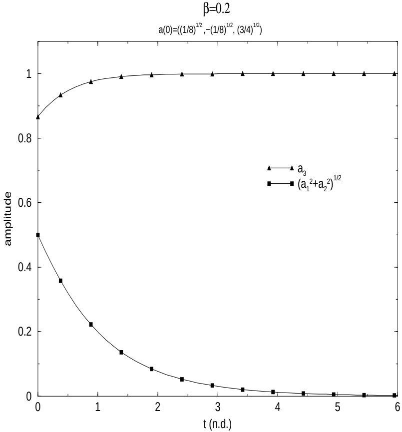
monotonic convergence

$$S = \begin{pmatrix} -0.4 & -0.4 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\tau_c}{\tau} = \frac{\gamma}{\tau(\lambda^J_{max} - \lambda^J_{int})} \approx 0.88$$

$$\tau = (E_{ij}E_{ij})^{-1/2} = 0.79$$

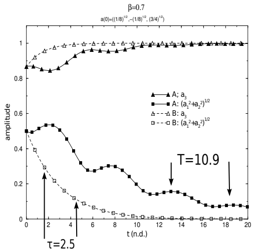
return fp-II



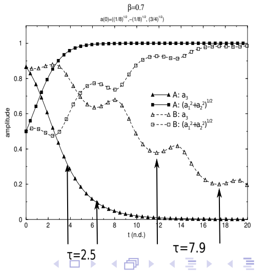
Complex eigenvalues: goals

- **Find the stable stationary solutions: fixed points and limit cycles.**
- **Characterize the different modes to approach the stationary solutions.**
- **Find characteristic time scales.**

fixed point



limit cycle



complex eigenvalues - Stationary solutions

limit-cycle

Fixed points and limit cycle

The eigenvalues of J

$$\lambda_1^J = \alpha$$
$$\lambda_{2,3}^J = -\frac{\alpha}{2} \pm i\omega$$

Stationary Solutions

- $\alpha > 0$ - **stable fixed point; unstable limit cycle**
- $\alpha < 0$ - **stable limit cycle; unstable fixed point**
- $\alpha = 0$ - **no attractor; periodic Jeffery Orbits**

complex eigenvalues - Limit cycles

Stability of the limit cycle

- **Γ is stable *iff* $\alpha < 0$.**
- **The period of rotation along Γ is $T = 2\pi\gamma/\omega$.**
- **If $\min_{x^0 \in \Gamma}(x^0, Ex^0) > \alpha$ the convergence to Γ is monotonic. Otherwise periodic growth and decay occur.**
- **The period average time scale of convergence is $\tau_c = \frac{2}{3}\frac{\gamma}{|\alpha|}$**

Example IV - convergence to a limit cycle - Contd.

fp IVa fp IVb

monotonic

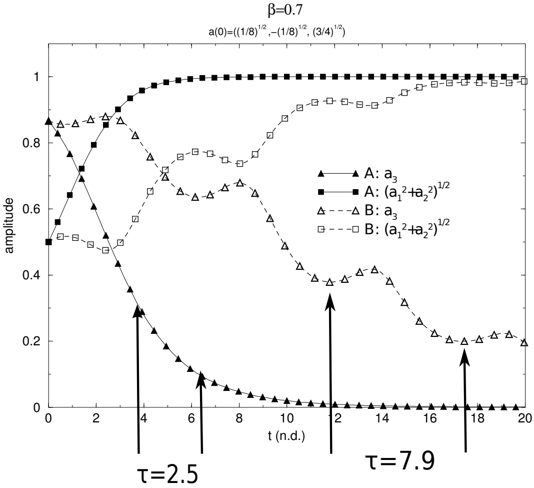
$\beta = 0.7$, $\gamma = 2.92$,
$\tau = 0.79$, $\tau_c/\tau \approx 2.5$,
$T/\tau \approx 43$.

$$S = \begin{pmatrix} 0.4 & 0.4 & 0 \\ 0. & 0.6 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

non-monotonic

$\beta = 0.7$, $\gamma = 2.92$,
$\tau = 0.41$ $\tau_c/\tau \approx 7.9$,
$T/\tau \approx 11$.

$$S = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.6 \end{pmatrix}$$



Orientation probability distribution function

- **The position of particles on the unit sphere is described by the unit vector x:**

$$\mathbf{x} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

- **The probability of finding a particle in the interval $[\theta, \theta + d\theta] \times [\phi, \phi + d\phi]$ at time t is given by**

$$f(\theta, \phi, t)\sin\theta d\theta d\phi$$

- **f satisfies the normalization condition:**

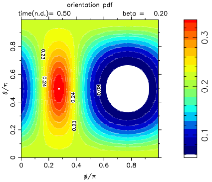
$$\int_0^\pi \int_{-\pi}^\pi f(\theta, \phi, t)\sin\theta d\theta d\phi = 1$$

- **f satisfies the Fokker-Planck equation**

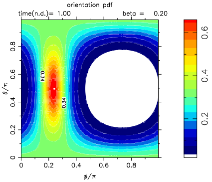
$$\frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) = 0$$
$$f|_{t=0} = f_0$$

Jeffery eq.

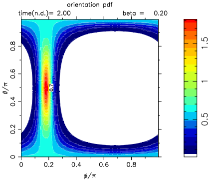
Fixed point: non-monotonic; example *I*; $\beta = 0.2$ example I



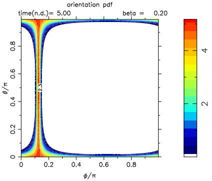
(a) t=0.5



(b) t=1

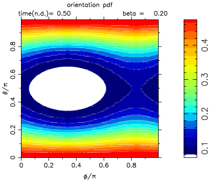


(c) t=2

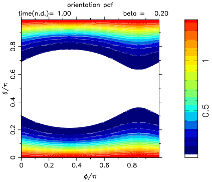


(d) t=5

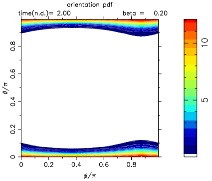
Fixed point: monotonic; example *II*; $\beta = 0.2$ example II



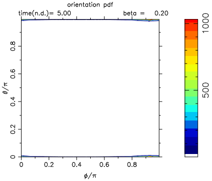
(e) t=0.5



(f) t=1

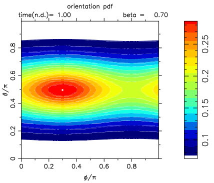


(g) t=2

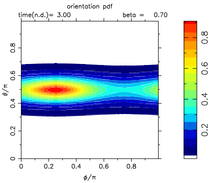


(h) t=5

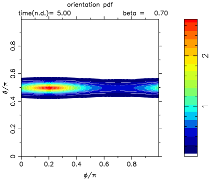
Limit cycle: monotonic; example *IVa*; $\beta = 0.7$ ▸ example IV



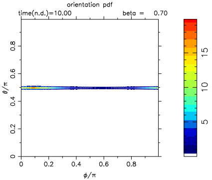
(i) t=1



(j) t=3



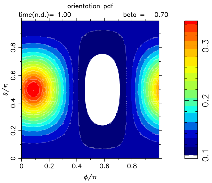
(k) t=5



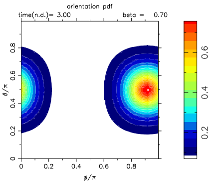
(l) t=10

Limit cycle: non-monotonic; example *IVb*; $\beta = 0.7$

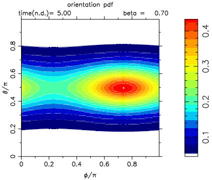
▸ example IV



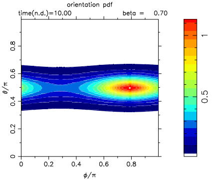
(m) t=1



(n) t=3



(o) t=5



(p) t=10

Summary

- The equation of motion of a single particle posseses three kinds of stationary solutions.
- For a given particle and flow field one of the following three cases may be realized:
  1. Three fixed points - only one is stable.
  2. One fixed points and one limit cycle - only one is stable.
  3. Periodic solutions.
- The "*choice*" *between the three cases is determined by the flow and the aspect ratio - as the particle becomes more spherical: case 1* $\longrightarrow$ *case 2.*
- *If the vorticity does not vanish convergence to a fixed point or limit cycle may be non-monotonic due to NON-NORMAL GROWTH.*
- *Convergence time to a fixed point or limit cycle may be long compared to the dissipation scale.*

Summary Contd.

- The Orientatiom p.d.f was computed numerically by solving the Fokker-Planck Equation.
- p.d.f behaviour:
  1. converges to a point - stable fixed point
  2. converges to a great circle - stable limit cycle
  3. periodic
- The convergence rate may be long compared to the dissipation rate.
- Although non-normal growth was obtained from local analysis it is observed over almost the entire orientation space.

Necessary cond. for no non normal growth

pert-jef

no vorticity - no non normal growth

Proposition:

a necessary condition for non normal growth is that the vorticity does not vanish.

Proof:

Suppose $R = 0$*, then* $J = E$ *is symmetric and all the eigenvectors are orthogonal.*

$\Longrightarrow U^1 = span\{X^2, X^3\}$ pert-jef

$\Longrightarrow \lambda_{max}^{P_{U^1}E} = \lambda_{max}^{P_{U^1}J} = max\{\lambda_2^J, \lambda_3^J\} < \lambda_1^J$

return