Similarity Solutions for Two-Phase Boundary-Layer Flows

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Similarity Solutions

- Reduce number of variables
- Consider sets of coordinates with similar properties
- Blasius
- Falkner & Skan

We define:

- A similarity variable:

$$\eta = y\left[\frac{Cx^{\alpha}(1+\alpha)}{2\nu x}\right]^{\frac{1}{2}}$$

- A stream function:

$$\Psi = \left(\frac{2\nu Cx^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{2}} f(\eta)$$

$$U = Cx^{\alpha}$$

$$u = \frac{\partial \Psi}{\partial y} = Uf'$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\nu}{x}\sqrt{\frac{\alpha+1}{2}\frac{Cx^{\alpha+1}}{\nu}}\left(f + \frac{\alpha-1}{\alpha+1}\eta f' \right)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$f''' + ff'' + \beta[1 - (f')^2] = 0,\ \beta = \frac{2\alpha}{1+\alpha}$$

- BC:
  - No-Slip : $f'(y=0) = f'(\eta=0) = 0$
  - Degree of Suction: $f(\eta=0) = f_0$
  - Velocity outside the BL (no over/under-shoot): $f'(\eta \to \infty) = 1,\ f\left(\eta < \eta_\infty\right) < 1$
  - For $\beta < -0.1984$, 3rd BC requires suction $\left(f_0 > 0\right)$

- Solution usually given for $0 \le \eta \le 5$
- But what if $\alpha < -1 \left(\equiv \beta > 2\right)$?
- Banks & Brodie: C<0, flow in direction of decreasing *x*
- But do we have to fiddle around with perspective and flow direction?
- Problems with non-linear nature

Stewartson, 1954
(My all-time favorite paper)

We note that if $\beta > 2$ there are in fact no solutions which

lead to real values of u and v so that this range need not be considered.

Are you sure about that?

- η would be positive, imaginary
- *f*' would still have to be real, increasing, so *f*, *f*" would have to be negative, imaginary.

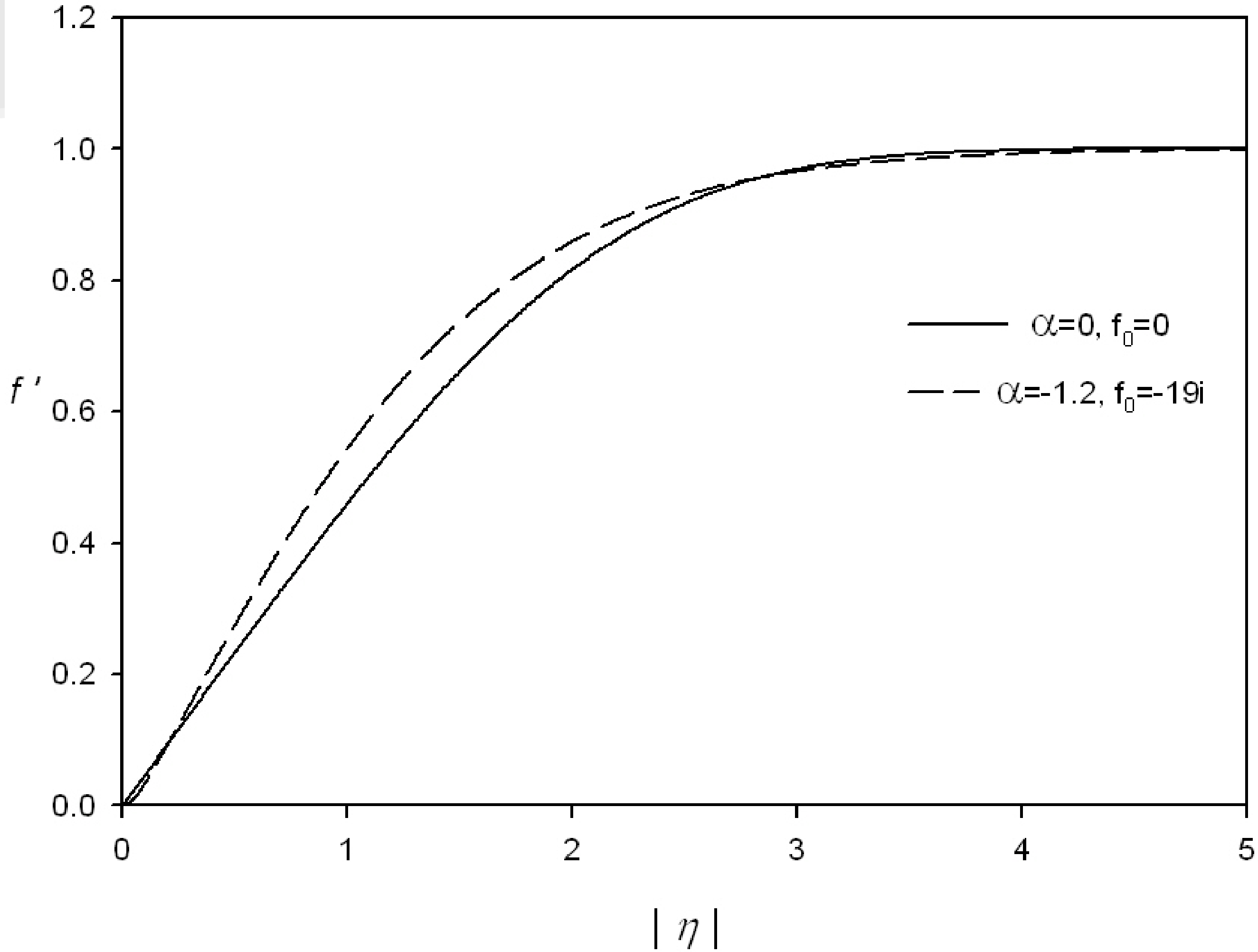
So what about the physical properties?

$$u = \frac{\partial \Psi}{\partial y} = Uf'$$

$$\frac{\eta_\infty}{\left[\mathrm{Re}_x(\alpha+1)\right]^{1/2}} = \frac{\delta}{x}$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\nu}{x}\sqrt{\frac{\alpha+1}{2}\frac{Cx^{\alpha+1}}{\nu}}\left(f + \frac{\alpha-1}{\alpha+1}\eta f'\right)$$

$$\Psi = \left(\frac{2\nu C x^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{2}} f(\eta)$$


1.2
1.0
0.8
0.6
0.4
0.2
0.0
f '
0
1
2
3
4
5
| η |
α=0, f0=0
α=-1.2, f0=-19i

- For -0.1984> β suction is necessary (proven by Stewartson)
- Can we prove the same for β>2?

Exact Solutions?

$$f''' + f\ f'' + \left(f'\right)^2 = 1$$

$$f'' + f\ f' = Y$$

$$f' + \frac{1}{2} f^2 = \frac{1}{2} Y^2 + 2a$$

$$Y = \eta + c; c = \sqrt{\left(f_0\right)^2 - 4a}$$

- For a=1/2 solution becomes exponential ("quickest") and thus:

$$f(Y) = Y + \frac{2\exp\left(-\frac{1}{2}Y^2\right)}{k + \int_Y \exp\left(-\frac{1}{2}S^2\right) dS}$$

$$k = \left(c + \sqrt{c^2 + 2}\right) \cdot \exp\left(-\frac{1}{2}c^2\right)$$

And the discrete phase?

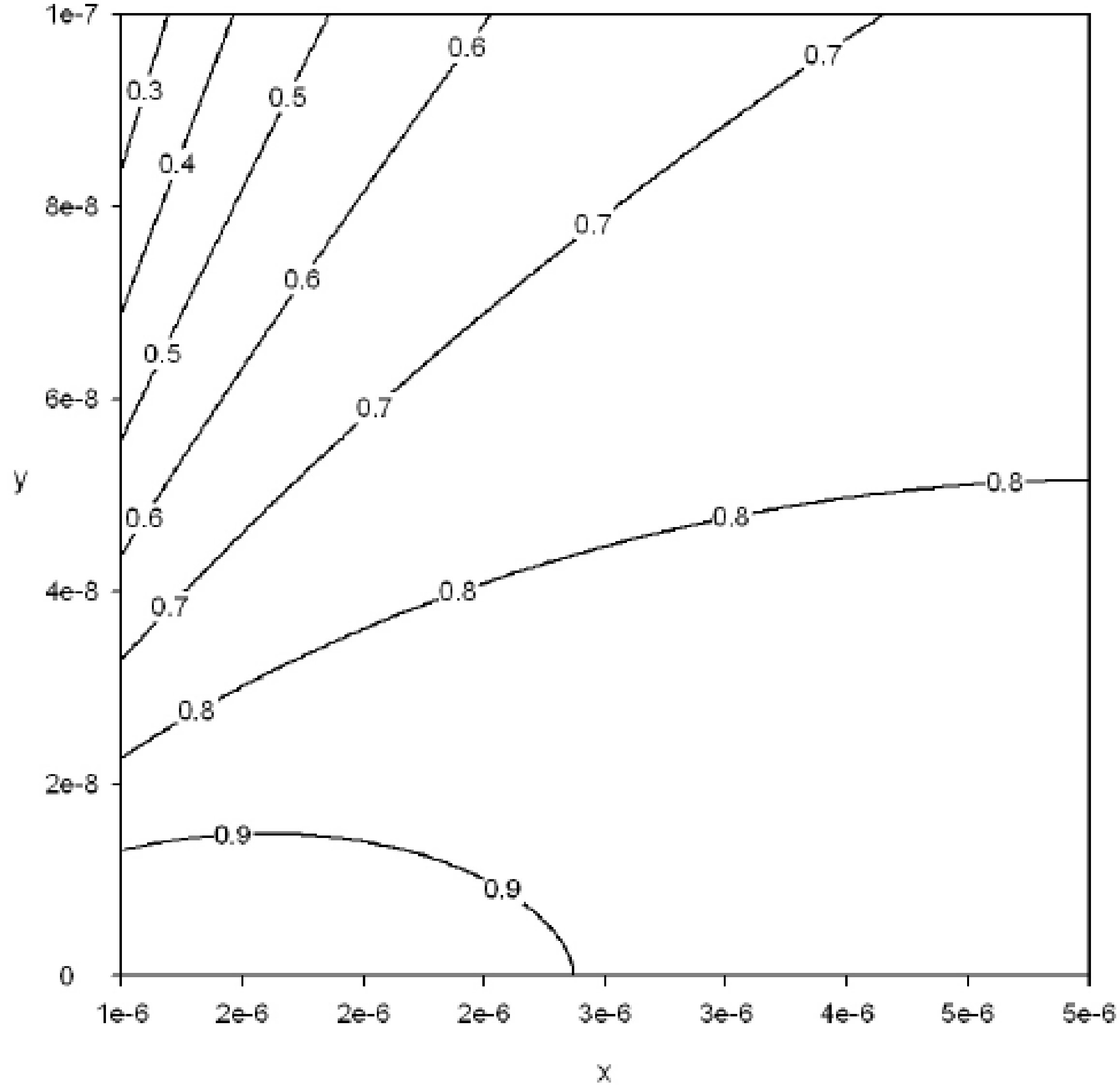
1. For "perfect convection":

$$u\frac{\partial Q}{\partial x}+v\frac{\partial Q}{\partial y}+Q\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]=0;$$

$$Q=q(\eta)\cdot x^{\delta}$$ (the equivalent of *f* ')

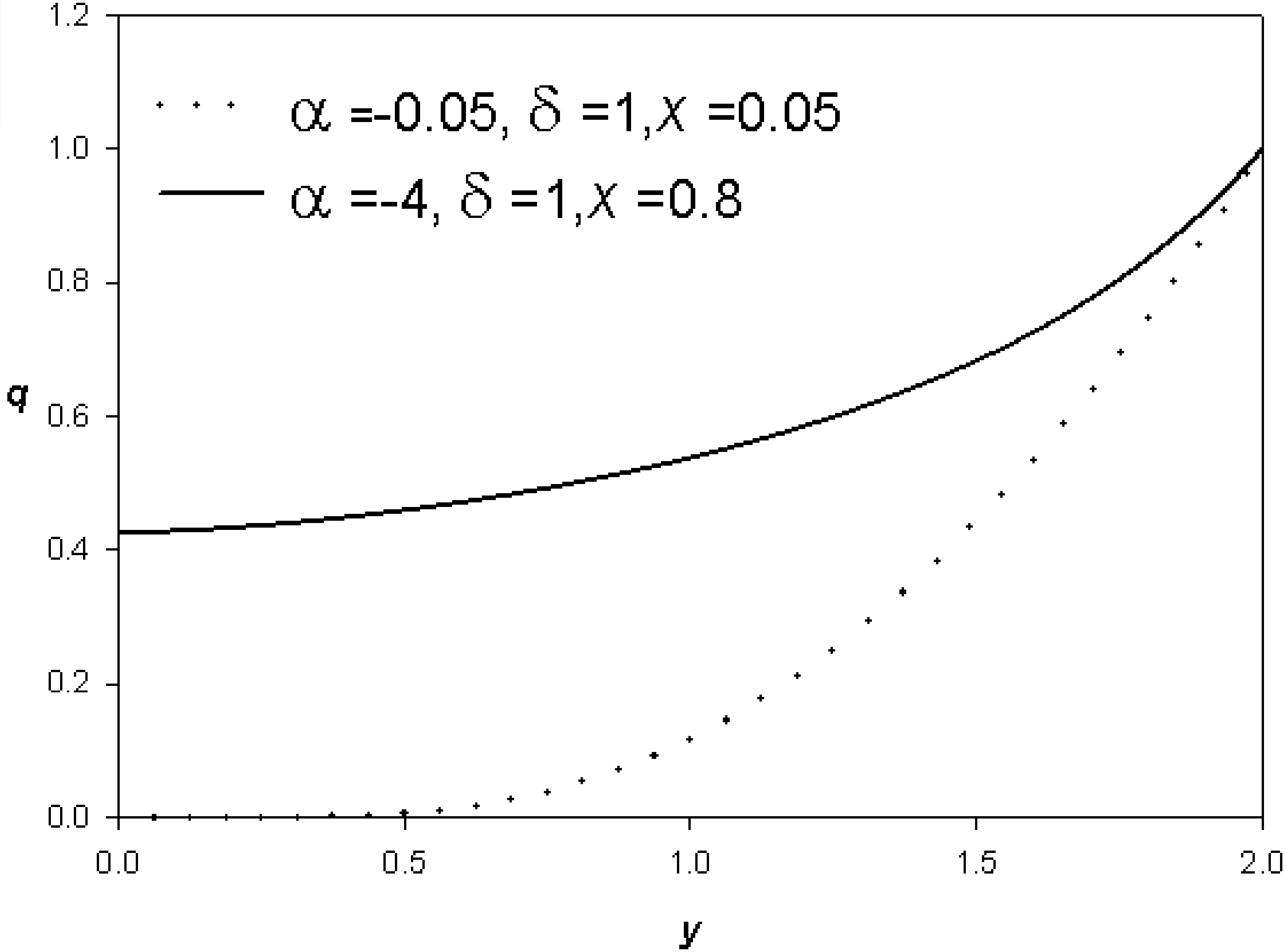
$$\Rightarrow q=C_1\cdot f^{\frac{2\delta}{\alpha+1}}$$

Concentration contours



$$q = C_1 \cdot f^{\frac{2\delta}{\alpha+1}}$$

- If $\alpha < -1$ then $C_1 = C_2 \cdot i^{\frac{2\delta}{\alpha+1}}$
- Two bifurcations: $\boxed{\delta = 0; \alpha = -1}$
- Delta bifurcates at a physically-relevant point
- But what's so special about -1?



α =-0.05, δ =1, χ =0.05
α =-4, δ =1, χ =0.8
q
y
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0

How about evaporation?

- Mass conservation of liquid in the spray

$$u\frac{\partial\tilde{Q}}{\partial x}+v\frac{\partial\tilde{Q}}{\partial y}=-\tilde{E}\,\tilde{Q}$$

$$\tilde{Q}=x^{\delta}\,Q(\eta)$$

$$\boxed{\tilde{E}=x^{\alpha-1}E}$$

$$\frac{2}{\alpha+1}\left(f'\delta+E\right)Q-f\,Q'=0$$

- And for α=-1/3 we may arrive at:

$$Q = Q_v \left(\frac{f}{f_v} \right)^{3\delta} \left(\frac{c}{\eta_v + c} \right)^{3E}$$

To conclude

- Imaginary values may extend the application of similarity solutions
- A closed analytical solution is presented for a particular value of β
- Weakness of method: limited by necessary assumptions, more stern as equation becomes more complex
- Don’t count the post office out!!!

“If you do not want Cabinet to spend too long discussing something, make it last on the agenda before lunch”

-- “Yes, minister”